

LACUNARY STATISTICAL QUASI-CAUCHY SEQUENCES

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ABSTRACT. In this paper we call a real-valued function lacunarily statistically ward continuous if it preserves lacunary statistical quasi-Cauchy sequences where a sequence (α_k) is defined to be lacunarily statistically quasi-Cauchy when the sequence $(\Delta\alpha_k)$ is lacunarily statistically convergent to 0. We prove theorems related to lacunary statistical ward compactness, statistical ward compactness, lacunary statistical ward continuity, statistical ward continuity, ward continuity, and uniform continuity. It turns out that any lacunarily statistically ward continuous function on a lacunary statistically ward compact subset A of \mathbf{R} is uniformly continuous on A . We also prove some inclusion theorems between the set of lacunary statistical quasi-Cauchy sequences and the set of statistical quasi-Cauchy sequences.

1. INTRODUCTION

A real function f is continuous if and only if, for each point α_0 in the domain, $\lim_{n \rightarrow \infty} f(\alpha_n) = f(\alpha_0)$ whenever $\lim_{n \rightarrow \infty} \alpha_n = \alpha_0$. This is equivalent to the statement that $(f(\alpha_n))$ is a convergent sequence whenever (α_n) is. This is also equivalent to the statement that $(f(\alpha_n))$ is a Cauchy sequence whenever (α_n) is provided that the domain of the function is either whole \mathbf{R} or a bounded and closed subset of \mathbf{R} where \mathbf{R} is the set of real numbers. These well known results for continuity for real functions in terms of sequences suggest to us to give a new type of continuity, namely, lacunary statistical ward continuity.

The purpose of this paper is to introduce new kinds of continuities defined via lacunary statistical quasi-Cauchy sequences, and investigate relations between some

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other kinds of continuities, such as ordinary continuity, uniform continuity, statistical continuity, lacunary statistical continuity.

2. PRELIMINARIES

We will use boldface letters \mathbf{x} , \mathbf{y} , \mathbf{z} , ... for sequences $\mathbf{p} = (p_n)$, $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$, ... of points in \mathbf{R} for the sake of abbreviation. s and c will denote the set of all sequences, and the set of convergent sequences of points in \mathbf{R} .

A subset of \mathbf{R} is compact if and only if it is closed and bounded. A subset A of \mathbf{R} is bounded if $|a| \leq M$ for all $a \in A$ where M is a positive real constant number. This is equivalent to the statement that any sequence of points in A has a Cauchy subsequence. The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to zero. Nevertheless, sequences which satisfy this weaker property are interesting in their own right. A sequence (α_n) of points in \mathbf{R} is quasi-Cauchy if $(\Delta\alpha_n)$ is a null sequence where $\Delta\alpha_n = \alpha_{n+1} - \alpha_n$. These sequences were named as quasi-Cauchy by Burton and Coleman [1], while they were called as forward convergent to zero sequences in [2],(see also [3], and [4]).

It is known that a sequence (α_n) of points in \mathbf{R} is slowly oscillating if

$$\lim_{\lambda \rightarrow 1^+} \overline{\lim}_n \max_{n+1 \leq k \leq [\lambda n]} |\alpha_k - \alpha_n| = 0$$

where $[\lambda n]$ denotes the integer part of λn . This is equivalent to the following if $\alpha_m - \alpha_n \rightarrow 0$ whenever $1 \leq \frac{m}{n} \rightarrow 1$ as, $m, n \rightarrow \infty$. Using $\varepsilon > 0$ s and δ s this is also equivalent to the case when for any given $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ and a positive integer $N = N(\varepsilon)$ such that $|\alpha_m - \alpha_n| < \varepsilon$ if $n \geq N(\varepsilon)$ and $n \leq m \leq (1+\delta)n$ (see [5]). Any Cauchy sequence is slowly oscillating, and any slowly oscillating sequence is quasi-Cauchy. There are quasi-Cauchy sequences which are not Cauchy. For example, the sequence (\sqrt{n}) is quasi-Cauchy, but Cauchy. Any subsequence of a Cauchy sequence is Cauchy. The analogous property fails for quasi-Cauchy sequences, and fails for slowly oscillating sequences as well. A counterexample for the case, quasi-Cauchy, is again the sequence $(a_n) = (\sqrt{n})$ with the subsequence $(a_{n^2}) = (n)$. A counterexample for the case slowly oscillating is the sequence $(\log_{10} n)$ with the subsequence (n) . Furthermore we give more examples without neglecting (see [6]): the sequences $(\sum_{k=1}^{\infty} \frac{1}{n})$, $(\ln n)$, $(\ln(\ln n))$, $(\ln(\ln(\ln n)))$ and

combinations like that are all slowly oscillating, but Cauchy. The bounded sequence $(\cos(6\log(n+1)))$ is slowly oscillating, but Cauchy. The sequences $(\cos(\pi\sqrt{n}))$ and $(\sum_{k=1}^{j=n} (\frac{1}{k})(\sum_{j=1}^k \frac{1}{j}))$ are quasi-Cauchy, but slowly oscillating (see also [7], [8], and [9]).

By a method of sequential convergence, or briefly a method, we mean an linear function G defined on a subspace of s , denoted by c_G , into \mathbf{R} . A sequence $\mathbf{x} = (x_n)$ is said to be G -convergent to ℓ if $\mathbf{x} \in c_G$ and $G(\mathbf{x}) = \ell$ (see [10]). In particular, \lim denotes the limit function $\lim \mathbf{x} = \lim_n x_n$ on the subspace c . A method G is called regular if $c \subset c_G$, i.e. every convergent sequence $\mathbf{x} = (x_n)$ is G -convergent with $G(\mathbf{x}) = \lim \mathbf{x}$.

Now we discuss some special classes of methods of sequential convergence that have been studied in the literature for real or complex number sequences. Firstly, for real and complex number sequences, we note that the most important transformation class is the class of matrix methods. Consider an infinite matrix $\mathbf{A} = (a_{nk})_{n,k=1}^{\infty}$ of real numbers. Then, for any sequence $\mathbf{x} = (x_n)$ the sequence \mathbf{Ax} is defined as

$$\mathbf{Ax} = (\sum_{k=1}^{\infty} a_{nk}x_k)_n$$

provided that each of the series converges. A sequence \mathbf{x} is called \mathbf{A} -convergent (or \mathbf{A} -summable) to ℓ if \mathbf{Ax} exists and is convergent with

$$\lim \mathbf{Ax} = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{nk}x_k = \ell.$$

Then ℓ is called the \mathbf{A} -limit of \mathbf{x} . We have thus defined a method of sequential convergence, i.e. $G(\mathbf{x}) = \lim \mathbf{Ax}$, called a matrix method or a summability matrix. A subset A of \mathbf{R} is G -sequentially compact if any sequence \mathbf{x} of points in A has a subsequence \mathbf{z} such that $G(\mathbf{z}) \in A$.

The Hahn-Banach theorem can be used to define methods which are not generated by a regular summability matrix. Banach used this theorem to show that the limit functional can be extended from the convergent sequences to the bounded sequences while preserving linearity, positivity and translation invariance; these extensions have come to be known as Banach limits. If a bounded sequence is assigned the same value ℓ by each Banach limit, the sequence is said to be almost convergent to ℓ . It is well known that a sequence $\mathbf{x} = (x_n)$ is almost convergent to ℓ if and

only if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n x_{k+j} = \ell,$$

uniformly in j . This defines a method of sequential convergence, i.e. $G(\mathbf{x})$:=almost limit of \mathbf{x} .

The idea of statistical convergence was formerly given under the name "almost convergence" by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [11]. The concept was formally introduced by Steinhaus [12] and Fast [13] and later was reintroduced by Schoenberg [14], and also independently by Buck [15]. Although statistical convergence was introduced over nearly the last seventy years, it has become an active area of research for twenty years. This concept has been applied in various areas such as number theory [16], measure theory [17], trigonometric series [11], summability theory [18], locally convex spaces [19], in the study of strong integral summability [20], turnpike theory [21], [22], [23], Banach spaces [24], and metrizable topological groups [25], and topological spaces [26], [27]. It should be also mentioned that the notion of statistical convergence has been considered, in other contexts, by several people like R.A. Bernstein, Z. Frolik, etc.

The concept of statistical convergence is a generalization of the usual notion of convergence that, for real-valued sequences, parallels the usual theory of convergence. A sequence (α_k) of points in \mathbf{R} is called statistically convergent to an element ℓ of \mathbf{R} if for each ε

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |\alpha_k - \ell| \geq \varepsilon\}| = 0,$$

and this is denoted by $st - \lim_{k \rightarrow \infty} \alpha_k = \ell$ [28] (see also [29]).

A sequence $\mathbf{p} = (p_n)$ of real numbers is called Abel convergent (or Abel summable) to ℓ if the series $\sum_{k=0}^{\infty} p_k x^k$ is convergent for $0 \leq x < 1$ and

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{k=0}^{\infty} p_k x^k = \ell.$$

In this case we write $Abel - \lim p_n = \ell$. Abel proved that if $\lim_{n \rightarrow \infty} p_n = \ell$, then $Abel - \lim p_n = \ell$, i.e. every convergent sequence is Abel convergent to the same limit ([30], see also [31], and [32]). As it is known that the converse is not always true in general, as we see that the sequence $((-1)^n)$ is Abel convergent to 0 but convergent in the ordinary sense.

Now we recall the definitions of ward compactness, and slowly oscillating compactness.

Definition 1. ([2]) A subset A of \mathbf{R} is called ward compact if whenever (α_n) is a sequence of points in A there is a quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (\alpha_{n_k})$ of (α_n) .

Definition 2. ([7], [9]) A subset A of \mathbf{R} is called slowly oscillating compact if whenever (α_n) is a sequence of points in A there is a slowly oscillating subsequence $\mathbf{z} = (z_k) = (\alpha_{n_k})$ of (α_n) .

In an unpublished work, Çakallı called a sequence (α_k) of points in \mathbf{R} statistically quasi-Cauchy if

$$st - \lim_{k \rightarrow \infty} \Delta \alpha_k = 0.$$

Any quasi-Cauchy sequence is statistically quasi-Cauchy, but the converse is not always true. Any statistically convergent sequence is statistically quasi-Cauchy. There are statistically quasi-Cauchy sequences which are not statistically convergent.

Definition 3. A subset A of \mathbf{R} is called statistically ward compact if whenever (α_n) is a sequence of points in A there is a statistical quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (\alpha_{n_k})$ of (α_n) .

Definition 4. A function defined on a subset A of \mathbf{R} is called statistically ward continuous if it preserves statistically quasi-Cauchy sequences, i.e. $(f(\alpha_n))$ is a statistically quasi-Cauchy sequence whenever (α_n) is.

By a lacunary sequence $\theta = (k_r)$, we mean an increasing sequence $\theta = (k_r)$ of positive integers such that $k_0 = 0$ and $h_r : k_r - k_{r-1} \rightarrow \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r . In this paper, we assume that $\liminf_r q_r > 1$.

The notion of lacunary statistical convergence was introduced, and studied by Fridy and Orhan in [33] and [34] (see also [35]) A sequence (α_k) of points in \mathbf{R} is called lacunary statistically convergent to an element ℓ of \mathbf{R} if

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |\alpha_k - \ell| \geq \varepsilon\}| = 0,$$

for every positive real number ε . The assumed condition a few lines above ensures the regularity of the lacunary statistical sequential method $G = S_\theta - \lim$.

Throughout the paper S , S_θ will denote the set of statistical convergent sequences, the set of lacunary statistical convergent sequences of points in \mathbf{R} , respectively.

Connor and Grosse-Erdmann [36] gave sequential definitions of continuity for real functions calling it G -continuity instead of A -continuity and their results cover the earlier works related to A -continuity. In particular, \lim denotes the limit function $\lim \mathbf{x} = \lim_n x_n$ on the subspace c , and $st - \lim$ denotes statistical limit function $st - \lim \mathbf{x} = st - \lim x_n$ on the subspace, S , and $S_\theta - \lim$ denotes lacunary statistical limit function $S_\theta - \lim \mathbf{x} = S_\theta - \lim x_n$ on the subspace, S_θ . We see that \lim , $st - \lim$, and $Abel - \lim$ are all regular methods without any restriction. $S_\theta - \lim$ is regular under the condition $\liminf_r q_r > 1$ which we assume throughout the paper. A function f is called G -sequentially continuous at $u \in \mathbf{R}$ provided that whenever a sequence \mathbf{x} of terms in \mathbf{R} , then the sequence $f(\mathbf{x}) = (f(x_n))$ is G -convergent to $f(u)$.

Recently, Çakalli gave a sequential definition of compactness, which is a generalization of ordinary sequential compactness, as in the following: a subset A of \mathbf{R} is G -sequentially compact if for any sequence (α_k) of points in A there exists a subsequence \mathbf{z} of the sequence such that $G(\mathbf{z}) \in A$. His idea enables us to obtain new kinds of compactness via most of the non-matrix sequential convergence methods, for example Abel method, as well as all matrix sequential convergence methods.

3. LACUNARY STATISTICAL QUASI-CAUCHY SEQUENCES

We call a sequence (α_k) of points in \mathbf{R} lacunarily statistically quasi-Cauchy if

$$S_\theta - \lim_{k \rightarrow \infty} \Delta \alpha_k = 0.$$

Any quasi-Cauchy sequence is lacunarily statistically quasi-Cauchy, but the converse is not always true. We see that a lacunarily statistically convergent sequence is lacunary statistically quasi-Cauchy. There are lacunarily statistically quasi-Cauchy sequences which are not lacunarily statistically convergent. Throughout the paper ΔS and ΔS_θ will denote the set of statistical quasi-Cauchy sequences and the set of lacunary statistical quasi-Cauchy sequences of points in \mathbf{R} , respectively.

Here we give some inclusion properties between the set of statistically quasi-Cauchy sequences and the set of lacunarily statistically quasi-Cauchy sequences.

Theorem 1. Let θ be any lacunary sequence. In order that $\Delta S \subset \Delta S_\theta$ it is necessary and sufficient that

$$\liminf q_r > 1.$$

Proof. Sufficiency. Let us first suppose that $\liminf q_r > 1$, $\liminf q_r = a$, say. We are going to prove that $\Delta S \subset \Delta S_\theta$. Write $b = \frac{a-1}{2}$. Then there exists a positive integer N such that $q_r \geq 1 + b$ for $r \geq N$. Since $h_r = k_r - k_{r-1}$, we have

$$\frac{h_r}{k_r} = 1 - \frac{k_{r-1}}{k_r} = 1 - \frac{1}{q_r} \geq 1 - \frac{1}{1+b} = \frac{b}{1+b}$$

for $r \geq N$. Let $(\alpha_k) \in \Delta S$. To prove that $(\alpha_k) \in \Delta S_\theta$ take any positive real number ε . Then for $r \geq n$ we have

$$\begin{aligned} \frac{1}{k_r} |\{k \leq k_r : |\Delta\alpha_k| \geq \varepsilon\}| &\geq \frac{1}{k_r} |\{k \in I_r : |\Delta\alpha_k| \geq \varepsilon\}| = \frac{h_r}{h_r k_r} |\{k \in I_r : |\Delta\alpha_k| \geq \varepsilon\}| \\ &\geq \frac{b}{1+b} \frac{1}{h_r} |\{k \in I_r : |\Delta\alpha_k| \geq \varepsilon\}| \end{aligned}$$

It follows from this inequality that $(\alpha_k) \in \Delta S_\theta$. Necessity. Now let us suppose that $\liminf q_r = 1$. Then we can choose a subsequence (k_{r_j}) of the lacunary sequence θ such that

$$\frac{k_{r_j}}{k_{r_j-1}} < 1 + \frac{1}{j}$$

and

$$\frac{k_{r_{j-1}}}{k_{r_{j-1}-1}} > j$$

where $r_j \geq r_{j-1} + 2$. Fix a positive real number c . Now define a sequence (α_k) by $\alpha_k = c + \frac{c+(-1)^k c}{2}$ if $k \in I_{r_j}$ for some $j = 1, 2, \dots, n, \dots$, and $\alpha_k = 0$ otherwise. Then the sequence (α_k) defined in this way is statistically quasi-Cauchy: for each m we can choose a positive integer j_m such that $k_{r_{j_m}} < m \leq k_{r_{j_m}+1}$. Then for each positive integer m

$$\begin{aligned} \frac{1}{m} |\{k \leq m : |\Delta\alpha_k| \geq c\}| &\leq \frac{1}{k_{r_{j_m}}} |\{k \leq k_{r_{j_m}} : |\Delta\alpha_k| \geq c\}| \\ &\leq \frac{1}{k_{r_{j_m}}} [|\{k \leq k_{r_{j_m}} : |\Delta\alpha_k| \geq \frac{c}{2}\}| + |\{k_{r_{j_m}} < k \leq m : |\Delta\alpha_k| \geq \frac{c}{2}\}|] \\ &\leq \frac{1}{k_{r_{j_m}}} |\{k \leq k_{r_{j_m}} : |\Delta\alpha_k| \geq \frac{c}{2}\}| + \frac{k_{r_{j_m}+1} - k_{r_{j_m}}}{k_{r_{j_m}}} < \frac{1}{j_m+1} + 1 + \frac{1}{j_m} - 1 < \frac{1}{j_m+1} + \frac{1}{j_m}. \end{aligned}$$

Thus $(\Delta\alpha_k) \in \Delta S$. Let us see that $(\alpha_k) \notin \Delta S_\theta$. We have

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{1}{h_{r_j}} |\{k_{r_{j-1}} < m \leq k_{r_j} : |\Delta\alpha_k| \geq \frac{c}{2}\}| &= \lim_{j \rightarrow \infty} \frac{1}{h_{r_j}} (k_{r_j} - k_{r_{j-1}}) = \\ \lim_{j \rightarrow \infty} \frac{1}{h_{r_j}} h_{r_j} &= 1, \text{ and } \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k_{r-1} \leq m < k_r : |\Delta\alpha_k| \geq \frac{c}{2}\}| \neq 0. \end{aligned}$$

This completes the proof of the theorem. \square

Theorem 2. Let θ be any lacunary sequence. In order that $\Delta S_\theta \subset \Delta S$ it is necessary and sufficient that

$$\limsup q_r < \infty.$$

Proof. Sufficiency. Suppose that $\limsup q_r < \infty$. Then we can find an $H > 0$ such that $q_r < H$ for any positive integer r . Let $(\alpha_k) \in \Delta S_\theta$. Take any positive real numbers c and ε . As (α_k) is a lacunary statistical quasi-Cauchy sequence, there exists a positive real number r_1 such that

$$\frac{2H}{h_r} |\{k \in I_r : |\Delta\alpha_k| \geq c\}| < \varepsilon$$

for any positive integer $r > r_1$. Now write

$$M = \max\{|\{k \in I_r : |\Delta\alpha_k| \geq c\}| : 1 \leq r \leq r_1\}$$

and let n be any positive integer satisfying $k_{r-1} < n \leq k_r$. Then for any positive integer n satisfying $k_{r-1} < n \leq k_r$ we have

$$\frac{1}{n} |\{k \leq n : |\Delta\alpha_k| \geq c\}| \leq r_1 M \frac{1}{k_{r-1}} + \frac{\varepsilon q_r}{2H}.$$

Since $k_r \rightarrow \infty$ as $r \rightarrow \infty$, there exists a positive integer r_2 such that

$$\frac{1}{k_{r_1-1}} < \frac{\varepsilon}{2r_1 M}$$

for $r > r_1$. Write $r_0 = \max\{r_1, r_2\}$. Hence for any positive integer n satisfying $k_{r-1} < n \leq k_r$ we obtain that

$$\frac{1}{n} |\{k \leq n : |\Delta\alpha_k| \geq c\}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $r > r_0$. It follows that $(\alpha_k) \in \Delta S$. Thus $\Delta S_\theta \subset \Delta S$.

Necessity. To prove that if $\Delta S_\theta \subset \Delta S$, then $\limsup q_r < \infty$, suppose that $\limsup q_r = \infty$. Let c be a fixed positive real number. Select a subsequence (k_{r_j}) of the lacunary sequence $\theta = (k_r)$ such that $q_{r_j} > j$, $k_{r_j} > j + 3$, and define a sequence (α_k) by $\alpha_k = c + \frac{c+(-1)^k c}{2}$ if $k_{r_j-1} < k \leq 2k_{r_j-1}$ for some $j = 1, 2, \dots$, and $\alpha_k = 0$ otherwise. Then for $j > 1$

$$\frac{1}{h_{r_j}} |\{k \leq k_{r_j} : |\Delta\alpha_k| \geq c\}| < \frac{k_{r_j-1}}{h_{r_j}} = \frac{k_{r_j-1}}{k_{r_j} - k_{r_j-1}} < \frac{1}{j-1}.$$

This implies that $(\alpha_k) \in \Delta S_\theta$. But $(\alpha_k) \notin \Delta S$. For

$$\frac{1}{2k_{r_j-1}} |\{k \leq 2k_{r_j-1} : |\Delta\alpha_k| \geq c\}| \geq \frac{k_{r_j-1}}{2k_{r_j-1}} = \frac{1}{2}$$

which implies that (α_k) is not statistically quasi-Cauchy. This completes the proof. \square

Combining Theorem 1 and Theorem 2 we have the following:

Corollary 3. Let θ be any lacunary sequence. $\Delta S_\theta = \Delta S$ if and only if

$$1 < \liminf q_r \leq \limsup q_r < \infty.$$

Corollary 4. Let θ be any lacunary sequence. $\Delta S_\theta = \Delta S$ if and only if $S = S_\theta$. The proof follows from Theorem 1 and Theorem 2 above, and Theorem 2 on page 116 in [25].

Now we give the definition of lacunary statistical ward compactness.

Definition 5. A subset A of \mathbf{R} is called lacunarily statistically ward compact if whenever (α_n) is a sequence of points in A there is a lacunary statistical quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (a_{n_k})$ of (α_n) .

We need the following lemmas for the proof of our theorems.

Lemma 5. Any lacunary statistical convergent sequence of points in \mathbf{R} with a statistical limit ℓ has a convergent subsequence with the same limit ℓ in the ordinary sense.

See Corollary 7 on page 118 of [25].

Theorem 6. A subset A of \mathbf{R} is bounded if and only if it is lacunarily statistically ward compact.

Proof. Although there is a proof in an unpublished work of Çakallı, we give a proof for completeness. Let A be any bounded subset of \mathbf{R} and (α_n) be any sequence of points in A . (α_n) is also a sequence of points in \overline{A} where \overline{A} denotes the closure of A . As \overline{A} is sequentially compact there is a convergent subsequence (α_{n_k}) of (α_n) (no matter the limit is in A or not). This subsequence is lacunary statistically convergent since lacunary statistical method is regular. Hence (α_{n_k}) is lacunary statistically quasi-Cauchy. Thus (a) implies (b). To prove that (b) implies (a), suppose that A is unbounded. If it is unbounded above, then one can construct a sequence (α_n) of numbers in A such that $\alpha_{n+1} > 1 + \alpha_n$ for each positive integer n . Then the sequence (α_n) does not have any lacunarily statistically quasi-Cauchy subsequence, so A is not lacunarily statistically ward compact. If A is bounded

above and unbounded below, then similarly we obtain that A is not lacunarily statistically ward compact. This completes the proof. \square

Corollary 7. A subset of \mathbf{R} is lacunary statistically ward compact if and only if it is lacunary statistically compact.

Corollary 8. A subset of \mathbf{R} is lacunary statistically ward compact if and only if it is statistically ward compact.

A sequence $\alpha = (\alpha_n)$ is δ -quasi-Cauchy if $\lim_{k \rightarrow \infty} \Delta^2 \alpha_n = 0$ where $\Delta^2 \alpha_n = a_{n+2} - 2a_{n+1} + \alpha_n$ ([37]). A subset A of \mathbf{R} is called δ -ward compact if whenever $\alpha = (\alpha_n)$ is a sequence of points in A there is a subsequence $\mathbf{z} = (z_k) = (\alpha_{n_k})$ of α with $\lim_{k \rightarrow \infty} \Delta^2 z_k = 0$. It follows from the above lemma that any lacunarily statistically ward compact subset of \mathbf{R} is δ -ward compact, and closure of a lacunarily statistically ward compact subset of \mathbf{R} is Abel sequentially compact.

We note that lacunarily statistically quasi-Cauchy sequences were studied in [38] in a different point of view.

We see that for any regular subsequential method G defined on \mathbf{R} , if a subset A of \mathbf{R} is G -sequentially compact, then it is lacunarily statistically ward compact. But the converse is not always true.

4. LACUNARY STATISTICAL WARD CONTINUITY

Now we give the definition of ward compactness in the following.

Definition 6. A function defined on a subset A of \mathbf{R} is called lacunarily statistically ward continuous if it preserves lacunarily statistically quasi-Cauchy sequences, i.e. $(f(\alpha_k))$ is a lacunarily statistically quasi-Cauchy sequence whenever (α_k) is.

We note that lacunary statistical ward continuity cannot be obtained by any sequential method G (see [39] for more information on G -continuity).

Composition of two lacunarily statistically ward continuous functions is lacunarily statistically ward continuous. Sum of two lacunarily statistically ward continuous functions is lacunarily statistically ward continuous, but product of lacunarily statistically ward continuous functions need not be lacunarily statistically ward continuous.

In connection with lacunary statistical quasi-Cauchy sequences and convergent sequences the problem arises to investigate the following types of continuity of functions on \mathbf{R} .

$$(\delta s_\theta): (\alpha_n) \in \Delta S_\theta \Rightarrow (f(\alpha_n)) \in \Delta S_\theta$$

$$(\delta s_\theta c): (\alpha_n) \in \Delta S_\theta \Rightarrow (f(\alpha_n)) \in c$$

$$(c): (\alpha_n) \in c \Rightarrow (f(\alpha_n)) \in c$$

$$(c\delta s_\theta): (\alpha_n) \in c \Rightarrow (f(\alpha_n)) \in \Delta S_\theta$$

$$(s_\theta): (\alpha_n) \in S_\theta \Rightarrow (f(\alpha_n)) \in S_\theta$$

We see that (δs_θ) is lacunary statistical ward continuity of f , (s_θ) is lacunary statistical continuity of f , and (c) is the ordinary continuity of f . It is easy to see that $(\delta s_\theta c)$ implies (δs_θ) , and (δs_θ) does not imply $(\delta s_\theta c)$; and (δs_θ) implies $(c\delta s_\theta)$, and $(c\delta s_\theta)$ does not imply (δs_θ) ; $(\delta s_\theta c)$ implies (c) and (c) does not imply $(\delta s_\theta c)$; and (c) is equivalent to $(c\delta s_\theta)$.

Now we give the implication (δs_θ) implies (s_θ) , i.e. any lacunarily statistically ward continuous function is lacunarily statistically continuous.

Theorem 9. If f is lacunarily statistically ward continuous on a subset A of \mathbf{R} , then it is lacunary statistically continuous on A .

Proof. Assume that f is a lacunarily statistically ward continuous function on a subset A of \mathbf{R} . Let (α_n) be any lacunarily statistically convergent sequence with $S_\theta - \lim_{k \rightarrow \infty} \alpha_k = \alpha_0$. Then the sequence

$$(\alpha_1, \alpha_0, \alpha_2, \alpha_0, \dots, \alpha_{n-1}, \alpha_0, \alpha_n, \alpha_0, \dots)$$

is lacunarily statistically convergent to α_0 . Hence it is lacunary statistically quasi-Cauchy. As f is lacunarily statistically ward continuous, the sequence

$$(f(\alpha_1), f(\alpha_0), f(\alpha_2), f(\alpha_0), \dots, f(\alpha_{n-1}), f(\alpha_0), f(\alpha_n), f(\alpha_0), \dots)$$

is lacunarily statistically quasi-Cauchy. It follows from this that the sequence $(f(\alpha_n))$ lacunarily statistically converges to $f(\alpha_0)$. This completes the proof of the theorem.

The converse is not always true for the function $f(x) = x^2$ is an example since the sequence (\sqrt{n}) is lacunary statistical quasi-Cauchy while $(f(\sqrt{n})) = (n)$ is not. \square

We state the following straightforward result related to ordinary continuity.

Corollary 10. If f is lacunarily statistically ward continuous, then it is continuous in the ordinary sense.

Corollary 11. If f is lacunarily statistically ward continuous, then it is statistically continuous.

Related to G -continuity we have much more in the following.

Corollary 12. If f is lacunarily statistically ward continuous, then it is G -continuous for any regular subsequential method G .

It is well known that any continuous function on a compact subset A of \mathbf{R} is uniformly continuous on A . It is also true for a regular subsequential method G that any lacunarily statistically ward continuous function on a G -sequentially compact subset A of \mathbf{R} is also uniformly continuous on A (see [6]). Furthermore, for lacunarily statistically ward continuous functions defined on a lacunarily statistically ward compact subset of \mathbf{R} , we have the following.

Theorem 13. Let A be a lacunarily statistically ward compact subset A of \mathbf{R} and let $f : A \rightarrow \mathbf{R}$ be a lacunarily statistically ward continuous function on A . Then f is uniformly continuous on A .

Proof. Suppose that f is not uniformly continuous on A so that there exists an $\varepsilon_0 > 0$ such that for any $\delta > 0$ $x, y \in E$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon_0$. For each positive integer n , fix $|\alpha_n - \beta_n| < \frac{1}{n}$, and $|f(\alpha_n) - f(\beta_n)| \geq \varepsilon_0$. Since A is lacunarily statistically ward compact, there exists a lacunary statistical quasi-Cauchy subsequence (α_{n_k}) of the sequence (α_n) . It is clear that the corresponding subsequence (β_{n_k}) of the sequence (β_n) is also lacunarily statistically quasi-Cauchy, since $(\beta_{n_{k+1}} - \beta_{n_k})$ is a sum of three lacunary statistical null sequences, i.e.

$$\beta_{n_{k+1}} - \beta_{n_k} = (\beta_{n_{k+1}} - \alpha_{n_{k+1}}) + (\alpha_{n_{k+1}} - \alpha_{n_k}) + (\alpha_{n_k} - \beta_{n_k}).$$

On the other hand it follows from the equality $\alpha_{n_{k+1}} - \beta_{n_k} = \alpha_{n_{k+1}} - \alpha_{n_k} + \alpha_{n_k} - \beta_{n_k}$ that the sequence $(\alpha_{n_{k+1}} - \beta_{n_k})$ is lacunarily statistically convergent to 0. Hence the sequence

$$(a_{n_1}, \beta_{n_1}, \alpha_{n_2}, \beta_{n_2}, \alpha_{n_3}, \beta_{n_3}, \dots, \alpha_{n_k}, \beta_{n_k}, \dots)$$

is lacunarily statistically quasi-Cauchy. But the transformed sequence

$$(f(\alpha_{n_1}), f(\beta_{n_1}), f(\alpha_{n_2}), f(\beta_{n_2}), f(\alpha_{n_3}), f(\beta_{n_3}), \dots, f(\alpha_{n_k}), f(\beta_{n_k}), \dots)$$

is not lacunarily statistically quasi-Cauchy. Thus f does not preserve lacunary statistical quasi-Cauchy sequences. This contradiction completes the proof of the theorem. \square

Corollary 14. If a function f is lacunarily statistically ward continuous on a bounded subset A of \mathbf{R} , then it is uniformly continuous on A .

Proof. The proof follows from the preceding theorem and Lemma 2. \square

Theorem 15. Lacunarily statistically ward continuous image of any lacunarily statistically ward compact subset of \mathbf{R} is lacunarily statistically ward compact.

Proof. Assume that f is a lacunarily statistically ward continuous function on a subset A of \mathbf{R} , and E is a lacunarily statistically ward compact subset of A . Let (β_n) be any sequence of points in $f(E)$. Write $\beta_n = f(\alpha_n)$ where $\alpha_n \in E$ for each positive integer n . lacunarily statistically ward compactness of E implies that there is a subsequence $(\gamma_k) = (\alpha_{n_k})$ of (α_n) with $S_\theta \lim_{k \rightarrow \infty} \Delta \gamma_k = 0$. Write $(t_k) = (f(\gamma_k))$. As f is lacunarily statistically ward continuous, $(f(\gamma_k))$ is lacunarily statistically quasi-Cauchy. Thus we have obtained a subsequence (t_k) of the sequence $(f(\alpha_n))$ with $S_\theta - \lim_{k \rightarrow \infty} \Delta t_k = 0$. Thus $f(E)$ is lacunarily statistically ward compact. This completes the proof of the theorem. \square

Corollary 16. Lacunarily statistically ward continuous image of any compact subset of \mathbf{R} is lacunarily statistically ward compact.

The proof follows from the preceding theorem.

Corollary 17. Lacunarily statistically ward continuous image of any bounded subset of \mathbf{R} is bounded.

The proof follows from Lemma 2 and Theorem 8.

Corollary 18. Lacunarily statistically ward continuous image of a G -sequentially compact subset of \mathbf{R} is lacunarily statistically ward compact for any regular sub-sequential method G .

It is a well known result that uniform limit of a sequence of continuous functions is continuous. This is also true in case of lacunary statistical ward continuity, i.e. uniform limit of a sequence of lacunarily statistically ward continuous functions is lacunarily statistically ward continuous.

Theorem 19. If (f_n) is a sequence of lacunarily statistically ward continuous functions on a subset A of \mathbf{R} and (f_n) is uniformly convergent to a function f , then f is lacunarily statistically ward continuous on A .

Proof. Let ε be a positive real number and (α_k) be any lacunarily statistically quasi-Cauchy sequence of points in A . By uniform convergence of (f_n) there exists a positive integer N such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in E$ whenever $n \geq N$. As f_N is lacunarily statistically ward continuous on A , we have

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f_N(\alpha_{k+1}) - f_N(\alpha_k)| \geq \frac{\varepsilon}{3}\}| = 0.$$

On the other hand we have

$$\begin{aligned} & \{k \in I_r : |f(\alpha_{k+1}) - f(\alpha_k)| \geq \varepsilon\} \subset \{k \in I_r : |f(\alpha_{k+1}) - f_N(\alpha_{k+1})| \geq \frac{\varepsilon}{3}\} \\ & \cup \{k \in I_r : |f_N(\alpha_{k+1}) - f_N(\alpha_k)| \geq \frac{\varepsilon}{3}\} \cup \{k \in I_r : |f_N(\alpha_k) - f(\alpha_k)| \geq \frac{\varepsilon}{3}\} \end{aligned}$$

Now it follows from this inclusion that

$$\begin{aligned} & \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f(\alpha_{k+1}) - f(\alpha_k)| \geq \varepsilon\}| \\ & \leq \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f(\alpha_{k+1}) - f_N(\alpha_{k+1})| \geq \frac{\varepsilon}{3}\}| + \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f_N(\alpha_{k+1}) - f_N(\alpha_k)| \geq \frac{\varepsilon}{3}\}| \\ & + \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f_N(\alpha_k) - f(\alpha_k)| \geq \frac{\varepsilon}{3}\}| = 0 + 0 + 0 = 0. \end{aligned}$$

This completes the proof of the theorem. \square

Theorem 20. The set of all lacunarily statistically ward continuous functions on a subset A of \mathbf{R} is a closed subset of the set of all continuous functions on A , i.e. $\overline{\Delta LSWC(A)} = \Delta LSWC(A)$ where $\Delta LSWC(A)$ is the set of all lacunarily statistically ward continuous functions on A , $\overline{\Delta LSWC(A)}$ denotes the set of all cluster points of $\Delta LSWC(A)$.

Proof. Let f be any element in the closure of $\Delta LSWC(A)$. Then there exists a sequence of points in $\Delta LSWC(A)$ such that $\lim_{k \rightarrow \infty} f_k = f$. To show that f is lacunarily statistically ward continuous take any lacunary statistical quasi-Cauchy sequence (α_k) of points in A . Let $\varepsilon > 0$. Since (f_k) converges to f , there exists an N such that for all $x \in A$ and for all $n \geq N$, $|f(x) - f_n(x)| < \frac{\varepsilon}{3}$. As f_N is lacunarily statistically ward continuous, we have

$$\lim_{n \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f_N(\alpha_{k+1}) - f_N(\alpha_k)| \geq \frac{\varepsilon}{3}\}| = 0.$$

On the other hand,

$$\begin{aligned} & \{k \in I_r : |f(\alpha_{k+1}) - f(\alpha_k)| \geq \varepsilon\} \subset \{k \in I_r : |f(\alpha_{k+1}) - f_N(\alpha_{k+1})| \geq \frac{\varepsilon}{3}\} \\ & \cup \{k \in I_r : |f_N(\alpha_{k+1}) - f_N(\alpha_k)| \geq \frac{\varepsilon}{3}\} \cup \{k \in I_r : |f_N(\alpha_k) - f(\alpha_k)| \geq \frac{\varepsilon}{3}\} \end{aligned}$$

Now it follows from this inclusion that

$$\begin{aligned} & \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f(\alpha_{k+1}) - f(\alpha_k)| \geq \varepsilon\}| \\ & \leq \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f(\alpha_{k+1}) - f_N(\alpha_{k+1})| \geq \frac{\varepsilon}{3}\}| + \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f_N(\alpha_{k+1}) - f_N(\alpha_k)| \geq \frac{\varepsilon}{3}\}| \\ & + \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |f_N(\alpha_k) - f(\alpha_k)| \geq \frac{\varepsilon}{3}\}| = 0 + 0 + 0 = 0. \end{aligned}$$

This completes the proof of the theorem. \square

Corollary 21. The set of all lacunarily statistically ward continuous functions on a subset A of \mathbf{R} is a complete subspace of the space of all continuous functions on A .

Proof. The proof follows from the preceding theorem. \square

5. CONCLUSION

In this paper, new types of continuities are introduced via lacunary statistical quasi-Cauchy sequences, and investigated. In the investigation we have obtained results related to lacunary statistical ward continuity, lacunary statistical continuity, statistical ward continuity, statistical continuity, G -sequential continuity, ordinary continuity, uniform continuity, lacunary statistical ward compactness, lacunary statistical compactness, statistical ward compactness, statistical compactness, G -sequential compactness, and ordinary compactness. We also proved some inclusion theorems between the set of lacunary statistical quasi-Cauchy sequences and the set of statistical quasi-Cauchy sequences.

For further study, we suggest to investigate lacunary statistical quasi-Cauchy sequences of fuzzy points, and lacunary statistical ward continuity for the fuzzy functions (see [40] for the definitions and related concepts in fuzzy setting). However due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work.

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